AM466/562: Finite Element Method
Solution of Homework 1

1. The finite element interpolant f_h of the function $f(x) = \sin(\pi x)$ is

$$f_h(x) = \sin(\pi/4)\phi_2(x) + \sin(\pi/2)\phi_3(x) + \sin(3\pi/4)\phi_4(x),$$

where ϕ_2, ϕ_3, ϕ_4 are the piece-wise linear finite element basis functions that represent the nodes $x = 1/4, 1/2, 3/4$, respectively. Their graphs are as follows

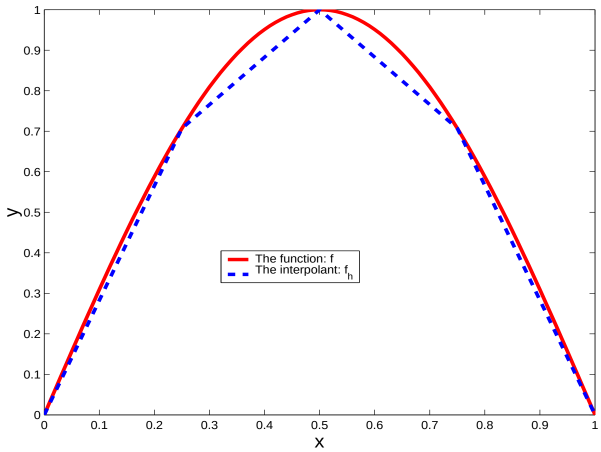

Figure 1: The finite element interpolant f_h of the function $f(x) = \sin(\pi x)$.

2. (a) Integrating the equation, we can easily find the exact solution

$$y = -\frac{1}{6}x^3 + c_1 x + c_2.$$

The boundary conditions $y(0) = y(1) = 0$ implies $c_1 = 1/6$ and $c_2 = 0$. Then

$$y = \frac{1}{6}x(1 - x^2).$$

(b) To derive the weak form of the problem, we multiply the equation by a sufficiently smooth *test* function $u(x)$:

$$-y''u = xu, \qquad u \in H_0^1(0, 1).$$

We then obtain the weak form by integrating over $[0,1]$:

$$-y'u|_0^1 + \int_0^1 y'u'dx = \int_0^1 xudx,$$

$$\int_0^1 y'u'dx = \int_0^1 xudx, \qquad u \in H_0^1(0,1). \tag{1}$$

(c) We seek the Galerkin approximation

$$y_h = \sum_{j=1}^{3} a_j \sin(j\pi x).$$

Then the stiffness matrix and load vector are

$$K_{ij} = \begin{cases} \int_0^1 \frac{d\sin(i\pi x)}{dx}\frac{d\sin(j\pi x)}{dx}dx = 0, & i \neq j, \\ \int_0^1 \frac{d\sin(i\pi x)}{dx}\frac{d\sin(j\pi x)}{dx}dx = \frac{i^2\pi^2}{2}, & i = j, \end{cases}$$

$$F_i = \int_0^1 x\sin(i\pi x)dx = \frac{-\cos(i\pi)}{i\pi}.$$

Solving the linear system

$$\sum_{j=1}^{3} K_{ij}a_j = F_i, \quad i = 1,2,3$$

we obtain the approximation

$$a_j = \frac{-2\cos(j\pi)}{j^3\pi^3},$$

$$y_h = \sum_{j=1}^{N} \frac{-2\cos(j\pi)}{j^3\pi^3}\sin(j\pi x).$$

Figure 2 shows that y_h is almost equal to y with $N = 10$.

(d) Let the nodes $x_e = (e-1)0.25$, $e = 1,2,3,4,5$ and ϕ_e be the basis function that represents the node x_e. Because the basis functions ϕ_1 and ϕ_5 do not satisfy the boundary condition, we can use only ϕ_2, ϕ_3 and ϕ_4 in our approximation. Therefore the finite element approximation has the following form

$$y_h(x) = a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x).$$

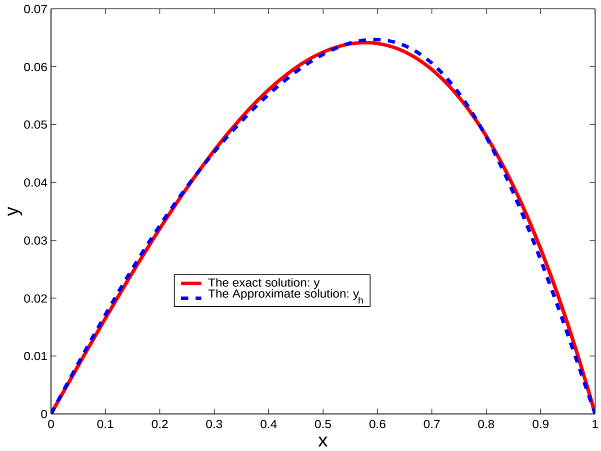

Figure 2: Fourier series approximation of the exact solution $y = \frac{1}{6}x(1-x^2)$.

Since

$$
\begin{aligned}
\frac{dN_1^e(x)}{dx} &= -\frac{1}{h} = -4, \quad \frac{dN_2^e(x)}{dx} = \frac{1}{h} = 4, \quad e = 1,2,3,4,\\
k_{ii}^e &= \int_{x_e}^{x_{e+1}} \frac{dN_i^e(x)}{dx}\frac{dN_i^e(x)}{dx}dx = \int_{x_e}^{x_{e+1}} 16dx = 4, \quad i = 1,2,\\
k_{12}^e &= k_{21}^e = \int_{x_e}^{x_{e+1}} \frac{dN_1^e(x)}{dx}\frac{dN_2^e(x)}{dx}dx = -\int_{x_e}^{x_{e+1}} 16dx = -4, \quad e = 2,3,\\
f_1^e &= \int_{x_e}^{x_{e+1}} xN_1^e(x)dx = \int_{x_e}^{x_{e+1}} \frac{x(x_{e+1}-x)}{h}dx = \frac{x_{e+1}+2x_e}{24}, \quad e = 2,3,4\\
f_2^e &= \int_{x_e}^{x_{e+1}} xN_2^e(x)dx = \int_{x_e}^{x_{e+1}} x\frac{x-x_e}{h}dx = \frac{2x_{e+1}+x_e}{24}, \quad e = 1,2,3,
\end{aligned}
$$

we have

Element e=1:

$$K^1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^1 = \begin{pmatrix} \frac{1}{48} \\ 0 \\ 0 \end{pmatrix}$$

Element e=2:

$$K^2 = \begin{pmatrix} 4 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^2 = \begin{pmatrix} \frac{1}{24} \\ \frac{1.25}{24} \\ 0 \end{pmatrix}$$

Element e=3:

$$K^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 0 \\ \frac{1.75}{24} \\ \frac{1}{12} \end{pmatrix}$$

Element e=4:

$$K^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad F^4 = \begin{pmatrix} 0 \\ 0 \\ \frac{2.5}{24} \end{pmatrix}$$

$$\begin{aligned} K &= K^1 + K^2 + K^3 + K^4 = \begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{pmatrix}, \\ F &= F^1 + F^2 + F^3 + F^4 = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{3}{16} \end{pmatrix} \end{aligned}$$

and then the system of equation is

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{3}{16} \end{pmatrix}.$$

Solving it, we obtain

$$a_2 = \frac{5}{128}, \quad a_3 = \frac{1}{16}, \quad a_4 = \frac{7}{128},$$

and then the finite element approximation y_h (see figure 3)

$$\begin{aligned} y_h(x) &= \frac{5}{128}\phi_2(x) + \frac{1}{16}\phi_3(x) + \frac{7}{128}\phi_4(x) \\ &= \begin{cases} \frac{20}{128}x, & 0 \le x \le 1/4, \\ \frac{20}{128}(0.5 - x) + \frac{1}{4}(x - 0.25), & 1/4 \le x \le 1/2, \\ \frac{1}{4}(0.75 - x) + \frac{28}{128}(x - 0.5), & 1/2 \le x \le 3/4, \\ \frac{28}{128}(1 - x), & 3/4 \le x \le 1. \end{cases} \end{aligned}$$

We note that in this particular problem $a_i = y(x_i)$, $i = 1, 2, 3, 4, 5$, where $y(x)$ is the exact solution.

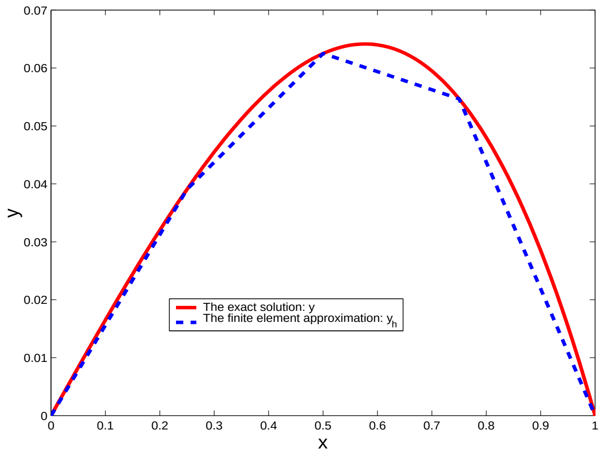

Figure 3: Finite element approximation of the exact solution $y = \frac{1}{6}x(1 - x^2)$.