# AM466: Finite Element Method
# Solution of Homework 2

Problem 5.

1. We first derive the weak form of the problem. We multiply the equation by a sufficiently smooth *test* function $u(x)$:

$$-y''u = f(x)u, \qquad u \in H^1(0,1).$$

We then obtain the weak form by integrating over $[0,1]$:

$$-y'u|_0^1 + \int_0^1 y'u'dx = \int_0^1 f(x)udx,$$

$$\int_0^1 y'u'dx = \int_0^1 f(x)udx + 2u(1) - u(0), \qquad u \in H^1(0,1). \tag{1}$$

Let the nodes $x_e = (e-1)0.5$, $e = 1,2,3$ and

$$\begin{aligned} p_1(\xi) &= \frac{1-\xi}{2} \quad -1 \le \xi \le 1, \\ p_2(\xi) &= \frac{1+\xi}{2} \quad -1 \le \xi \le 1, \\ x &= \frac{1}{2}(0.5\xi + x_e + x_{e+1}). \end{aligned}$$

Then the entries of the element stiffness matrix are

$$\begin{aligned} k_{11}^e &= \int_{-1}^1 \frac{d}{d\xi}(p_1(\xi))\frac{d}{d\xi}(p_1(\xi))\frac{2}{0.5}d\xi = \int_{-1}^1 \frac{1}{4}\cdot 4 = 2, \\ k_{12}^e &= k_{21}^e = \int_{-1}^1 \frac{d}{d\xi}(p_1(\xi))\frac{d}{d\xi}(p_2(\xi))\frac{2}{0.5}d\xi = \int_{-1}^1 \frac{-1}{4}\cdot 4 = -2, \\ k_{22}^e &= \int_{-1}^1 \frac{d}{d\xi}(p_2(\xi))\frac{d}{d\xi}(p_2(\xi))\frac{2}{0.5}d\xi = \int_1^1 \frac{1}{4}\cdot 4 = 2, \end{aligned}$$

and the local load vector is

$$\begin{aligned} f_1^e &= \int_{-1}^1 f(x(\xi))p_1(\xi)\frac{0.5}{2}d\xi = \int_{-1}^1 f(x(\xi))\frac{1-\xi}{2}\frac{1}{4}d\xi, \\ f_2^e &= \int_{-1}^1 f(x(\xi))p_2(\xi)\frac{0.5}{2}d\xi = \int_{-1}^1 f(x(\xi))\frac{1+\xi}{2}\frac{1}{4}d\xi. \end{aligned}$$

Thus the system of equation is

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1^1 - 1 \\ f_2^1 + f_1^2 \\ f_2^2 + 2 \end{pmatrix}. \tag{2}$$

Since the determinant of the stiffness matrix is zero, it is singular.

2. Taking $u = 1$ in (1), we obtain

$$0 = \int_0^1 f(x)dx + 2 - 1,$$

and so

$$\int_0^1 f(x)dx = -1,$$

3. Setting $y_1 = 0$, we reduce the system (2) to

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_2^1 + f_1^2 \\ f_2^2 + 2 \end{pmatrix}, \tag{3}$$

which is invertible. Moreover, from the first equation of (2), we derive the equation for $y_2$

$$-2y_2 = f_1^1 - 1. \tag{4}$$

4. Solving (3), we obtain

$$y_2 = \frac{1}{2}(f_2^1 + f_1^2 + f_2^2 + 2).$$

Solving (4), we obtain

$$y_2 = \frac{1}{2}(1 - f_1^1).$$

They are equal because

$$f_2^1 + f_1^2 + f_2^2 + 2 - 1 + f_1^1 = \int_0^1 f(x)dx + 1 = 0.$$