AM466: Finite Element Method

Solutions of Homework 4

Problem 5. Consider the Poisson's equation

$$\begin{aligned}
\frac{\partial}{\partial x}\left((1+x^2)\frac{\partial \Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left((1+y^2)\frac{\partial \Phi}{\partial y}\right)-4xy &= 0, \quad \text{in} \quad \Omega,\\
(1+x^2)\frac{\partial \Phi}{\partial x}n_x+(1+y^2)\frac{\partial \Phi}{\partial y}n_y &= -y, \quad x=0,\\
(1+x^2)\frac{\partial \Phi}{\partial x}n_x+(1+y^2)\frac{\partial \Phi}{\partial y}n_y &= 65y, \quad x=8,\\
\Phi &= 0, \quad y=0,\\
\Phi &= 4x, \quad y=4.
\end{aligned}$$

1. (10 marks) Derive the weak form of the above boundary value problem.

   Solution. Multiplying the equation by a *test* function $\Psi(x) \in H^1(\Omega)$ and Integrating over Ω yields

$$\int_\Omega \left[\Psi\frac{\partial}{\partial x}\left((1+x^2)\frac{\partial \Phi}{\partial x}\right)+\Psi\frac{\partial}{\partial y}\left((1+y^2)\frac{\partial \Phi}{\partial y}\right)-4xy\Psi\right]dxdy=0.$$

Integration by parts gives

$$\begin{aligned}
&\int_\Omega \left[(1+x^2)\frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial x} + (1+y^2)\frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial y}\right] dxdy \\
= \quad & -4\int_\Omega xy\Psi dxdy + \int_S \Psi\left[(1+x^2)\frac{\partial\Phi}{\partial x}n_x + (1+y^2)\frac{\partial\Phi}{\partial y}n_y\right] dS. \qquad (1)
\end{aligned}$$

Since $\Phi = \Phi^*$ on the surfaces $y = 0, 4$ is known, Ψ can be taken such that $\Psi = 0$ there and then

$$\begin{aligned}
&\int_\Omega \left[(1+x^2)\frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial x} + (1+y^2)\frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial y}\right] dxdy \\
= \quad & -4\int_\Omega xy\Psi dxdy - \int_0^4 \Psi(0,y)ydy + 65\int_0^4 \Psi(8,y)ydy. \qquad (2)
\end{aligned}$$

Therefore the weak form for the problem is:

Find $\Phi \in H^1(\Omega)$ such that $\Phi = 0$ on $y = 0, \Phi = 4x$ on $y = 4,$ and (2) holds for all

$\Psi \in H^1(\Omega)$ such that $\Psi = 0$ on $y = 0, 4$.

2. (10 marks) Write down the shape functions of the element 3 that represent the nodes 6, 2, 7 respectively.

Solution. Since the equation for the line through the nodes 2 and 7 is $x = 2$, the shape functions of the element 3 that represent the nodes 6 is

$$N_1^3(x,y) = \frac{2-x}{2}.$$

Since the equation for the line through the nodes 6 and 7 is $y = 2$, the shape functions of the element 3 that represent the nodes 2 is

$$N_2^3(x,y) = \frac{2-y}{2}.$$

Since the equation for the line through the nodes 6 and 2 is $x + y = 2$, the shape functions of the element 3 that represent the nodes 7 is

$$N_3^3(x,y) = \frac{x+y+2}{2}.$$

3. (10 marks) Calculate the local element stiffness matrix (k_{ij}^3), $(i, j = 1, 2, 3)$.

Solution.

$$
\begin{aligned}
k_{11}^3 &= \int_\Omega \left[(1+x^2)\frac{\partial N_1^3}{\partial x}\frac{\partial N_1^3}{\partial x} + (1+y^2)\frac{\partial N_1^3}{\partial y}\frac{\partial N_1^3}{\partial y}\right] dxdy \\
&= \frac{1}{4}\int_0^2 dx \int_{2-x}^2 (1+x^2)dy \\
&= \frac{3}{2},
\end{aligned}
$$

$$
\begin{aligned}
k_{12}^3 &= k_{21}^3 = \int_\Omega \left[(1+x^2)\frac{\partial N_1^3}{\partial x}\frac{\partial N_2^3}{\partial x} + (1+y^2)\frac{\partial N_1^3}{\partial y}\frac{\partial N_2^3}{\partial y}\right] dxdy \\
&= \int_0^2 dx \int_{2-x}^2 0dxdy \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
k_{13}^3 &= k_{31}^3 = \int_\Omega \left[(1+x^2)\frac{\partial N_1^3}{\partial x}\frac{\partial N_3^3}{\partial x} + (1+y^2)\frac{\partial N_1^3}{\partial y}\frac{\partial N_3^3}{\partial y}\right] dxdy \\
&= -\frac{1}{4}\int_0^2 dx \int_{2-x}^2 (1+x^2)dy \\
&= -\frac{3}{2},
\end{aligned}
$$

$$
\begin{aligned}
k_{22}^3 &= \int_\Omega \left[(1+x^2)\frac{\partial N_2^3}{\partial x}\frac{\partial N_2^3}{\partial x} + (1+y^2)\frac{\partial N_2^3}{\partial y}\frac{\partial N_2^3}{\partial y}\right] dxdy \\
&= \frac{1}{4}\int_0^2 dx \int_{2-x}^2 (1+y^2)dy \\
&= \frac{3}{2},
\end{aligned}
$$

$$
\begin{aligned}
k_{23}^3 &= k_{32}^3 = \int_\Omega \left[(1+x^2)\frac{\partial N_2^3}{\partial x}\frac{\partial N_3^3}{\partial x} + (1+y^2)\frac{\partial N_2^3}{\partial y}\frac{\partial N_3^3}{\partial y}\right] dxdy \\
&= -\frac{1}{4}\int_0^2 dx \int_{2-x}^2 (1+y^2)dy \\
&= -\frac{3}{2},
\end{aligned}
$$

$$
\begin{aligned}
k_{33}^3 &= \int_\Omega \left[(1+x^2)\frac{\partial N_3^3}{\partial x}\frac{\partial N_3^3}{\partial x} + (1+y^2)\frac{\partial N_3^3}{\partial y}\frac{\partial N_3^3}{\partial y}\right] dxdy \\
&= \frac{1}{4}\int_0^2 dx \int_{2-x}^2 (2+x^2+y^2)dy \\
&= 3.
\end{aligned}
$$

4. (10 marks) Calculate the surface integrals on the boundary $x = 0$ and $x = 8$.

Solution. We need to compute $\Sigma_6^1, \Sigma_{11}^1, \Sigma_1^4, \Sigma_6^4, \Sigma_{10}^{13}, \Sigma_{15}^{13}, \Sigma_5^{15}, \Sigma_{10}^{15}$ associated with the elements 1,4, 13, 15. The shape function on the element 13 that represents the node 10 is

$$N_1^{13}(x, y) = 2 - \frac{y}{2}.$$

So

$$\Sigma_{10}^{13} = \int_2^4 (2 - \frac{y}{2})(65y)dy = 520/3.$$

The shape function on the element 13 that represents the node 15 is

$$N_2^{13}(x, y) = \frac{x + y - 10}{2}.$$

Because its restriction on $x = 8$ is $-1 + y/2$, we have

$$\Sigma_{15}^{13} = \int_2^4 (\frac{y}{2} - 1)(65y) = 650/3.$$

The shape function on the element 15 that represents the node 5 is

$$N_1^{15}(x, y) = 1 - \frac{y}{2}.$$

So

$$\Sigma_5^{15} = \int_0^2 (1 - \frac{y}{2})(65y)dy = 130/3.$$

The shape function on the element 15 that represents the node 10 is

$$N_2^{15}(x, y) = \frac{x + y - 8}{2}.$$

Because its restriction on $x = 8$ is $y/2$, we have

$$\Sigma_{10}^{15} = \int_0^2 \frac{y}{2}(65y)dy = 260/3.$$

The shape function on the element 1 that represents the node 6 is

$$N_1^1(x, y) = 2 - \frac{y}{2}.$$

So

$$\Sigma_6^1 = \int_2^4 (2 - \frac{y}{2})(-y)dy = -8/3.$$

The shape function on the element 1 that represents the node 11 is

$$N_2^1(x,y) = \frac{y-x-2}{2}.$$

Because its restriction on $x=0$ is $-1+y/2$, we have

$$\Sigma_{11}^1 = \int_2^4 (\frac{y}{2}-1)(-y) = -10/3.$$

The shape function on the element 4 that represents the node 1 is

$$N_1^4(x,y) = \frac{2-x-y}{2}.$$

So

$$\Sigma_1^4 = \int_0^2 (1-\frac{y}{2})(-y)dy = -2/3.$$

The shape function on the element 4 that represents the node 6 is

$$N_2^4(x,y) = \frac{y}{2}.$$

So we have

$$\Sigma_6^4 = \int_0^2 \frac{y}{2}(-y)dy = -4/3.$$

5. (10 marks) Use Poisson.m to find the finite element approximate solution of the boundary value problem. Plot the approximation and the exact solution $\Phi = xy$.

   Solution. The NPcode, INITIAL.m, and COEF.m are modifies as follows:

```
% XORD   YORD   NPcode
%-----------------------------
     0      0      41
     2      0      1
     4      0      1
     6      0      1
     8      0      21
     0      2      42
     2      2      0
     4      2      0
     6      2      0
     8      2      22
     0      4      43
     2      4      3
```

```
        4      4      3
        6      4      3
        8      4      23
  %----------------------------

%--------------------------------
%     INITIAL.m
%
%     Specify boundary conditions
%---------------------------------------------

     for I=1:NUMNP
        if NPcode(I) == 1
           NPBC(I) = 1;
           PHI(I)  = 0;
           Q(I)    = 0;
        elseif NPcode(I) == 21
           NPBC(I) = 0;
           PHI(I)  = 0;
           Q(I)    = 130/3;
        elseif NPcode(I) == 22
           NPBC(I) = 0;
           PHI(I)  = 0;
           Q(I)    = 780/3;
        elseif NPcode(I) == 23
           NPBC(I) = 0;
           PHI(I)  = 0;
           Q(I)    = 650/3;
        elseif NPcode(I) == 3
           NPBC(I) = 1;
           PHI(I)  = 4*XORD(I);
           Q(I)    = 0;
        elseif NPcode(I) == 41
           NPBC(I) = 0;
           PHI(I)  = 0;
           Q(I)    = -2/3;
        elseif NPcode(I) == 42
           NPBC(I) = 0;
           PHI(I)  = 0;
           Q(I)    = -4;
        elseif NPcode(I) == 43
```

```
            NPBC(I) = 0;
            PHI(I)  = 0;
            Q(I)    = -10/3;
         end
      end

%====================================
%
%    Coefficients for poisson.m
%
%     RXI = k_x
%     RYI = k_Y
%     QVI = Q
%
%===================================
         RXI = 1+XC^2;
         RYI = 1+YC^2;
         QVI = -4*XC*YC;
```

The exact solution $\Phi = xy$ and its finite element approximation are shown in Figure 1.

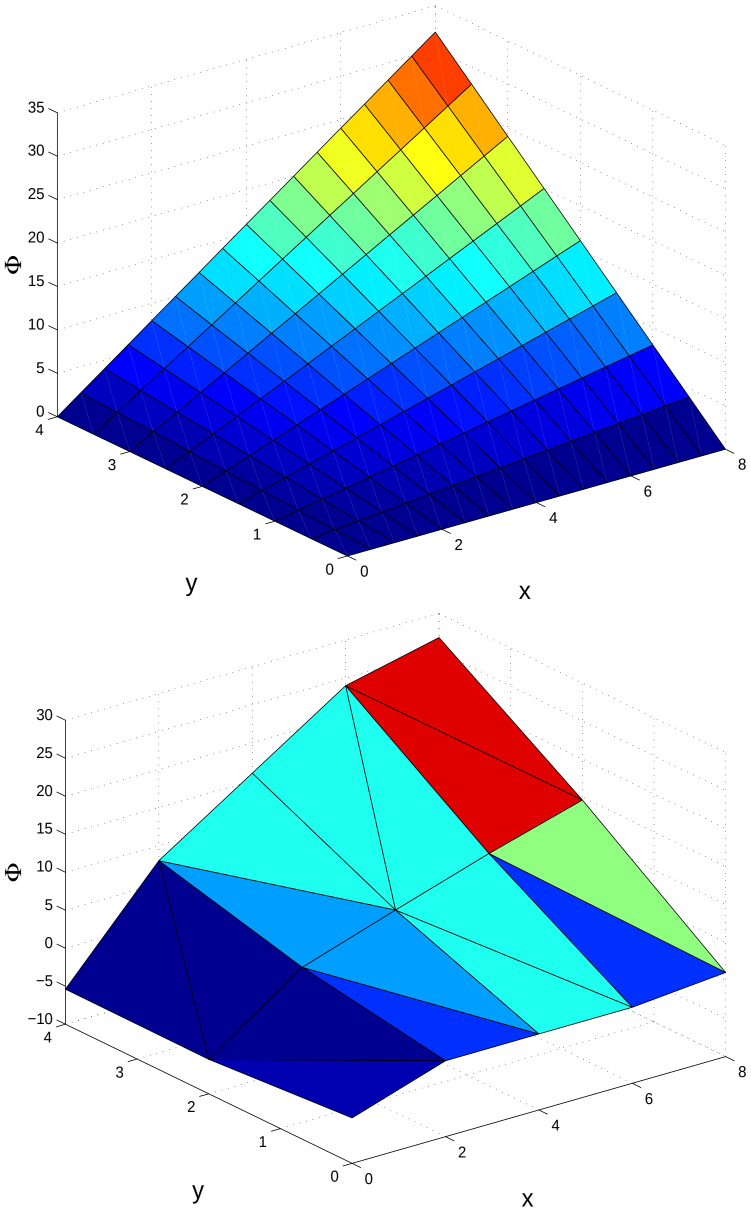

Figure 1: The above: Exact solution $\Phi = xy$; The below: The finite element approximation.