Review of One Dimensional Problems

- **What have I taught?**

  1. The stretched wire model: second-order linear ordinary differential equations (strong form)

  $$\frac{d}{dx}\left(A(x)\frac{dy}{dx}\right) + B(x)\frac{dy}{dx} + C(x)y + D(x) = 0, \quad 0 \le x \le L, \tag{1}$$

  with one of the boundary conditions

  $$\text{Dirichlet condition:} \quad y(0) = y_0, \quad y(L) = y_L, \tag{2}$$

  $$\text{Neumann condition:} \quad A(0)\frac{dy}{dx}(0) = -q_0, \quad A(L)\frac{dy}{dx}(L) = q_L. \tag{3}$$

  2. The weak form:

     – For the Dirichlet boundary condition, find a function $y \in H^1(0,1)$ such that

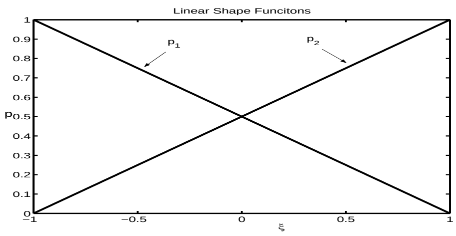
  $$\begin{aligned} &\int_0^L \left(A(x)\frac{dy}{dx}\right)\frac{du}{dx}dx - \int_0^L B(x)\frac{dy}{dx}udx - \int_0^L C(x)yudx \\ = &\int_0^L D(x)udx, \quad \text{for all } u \in H_0^1(0,1). \end{aligned}$$

     – For the Neumann boundary condition, find a function $y \in H^1(0,1)$ such that

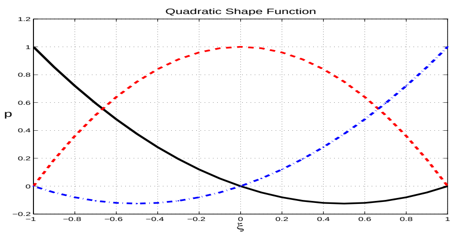
  $$\begin{aligned} &\int_0^L \left(A(x)\frac{dy}{dx}\right)\frac{du}{dx}dx - \int_0^L B(x)\frac{dy}{dx}udx - \int_0^L C(x)yudx \\ = &\int_0^L D(x)udx + q_0u(0) + q_Lu(L), \quad \text{for all } u \in H^1(0,1). \end{aligned}$$

3. Linear and quadratic shape functions:

$$p_1(\xi) = \frac{1-\xi}{2}, \quad p_2(\xi) = \frac{1+\xi}{2}.$$



$$p_1(\xi) = \frac{\xi(\xi-1)}{2}, \quad p_2(\xi) = 1-\xi^2, \quad p_3(\xi) = \frac{\xi(1+\xi)}{2}.$$



4. Galerkin's Approximations:

$$y_h = \sum_{j=1}^{N+1} y_j \phi_j(x).$$

$$\sum_{j=1}^{N+1} K_{ij} y_j = F_i + \Sigma_i, \qquad i = 1, \ldots, N+1, \tag{4}$$

$$\begin{aligned} K_{ij} &= \int_0^L \left(A(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} - B(x) \frac{d\phi_j(x)}{dx} \phi_i(x) - C(x)\phi_i(x)\phi_j(x) \right) dx \\ F_i &= \int_0^L D(x)\phi_i(x) dx, \\ \Sigma_i &= q_0 \phi_i(0) + q_L \phi_i(L). \end{aligned}$$

5. Calculations of element matrices:

$$\begin{aligned}
k^e_{mn} &= \int_{-1}^{1} A(x(\xi))\frac{d}{d\xi}(p_m(\xi))\frac{2}{h_e}\frac{d}{d\xi}(p_n(\xi))\frac{2}{h_e}\frac{h_e}{2}d\xi \\
&\quad - \int_{-1}^{1} B(x(\xi))p_m(\xi)\frac{d}{d\xi}(p_n(\xi))\frac{2}{h_e}\frac{h_e}{2}d\xi \\
&\quad - \int_{-1}^{1} C(x(\xi))p_m(\xi)p_n(\xi)\frac{h_e}{2}d\xi, \\
f^e_m &= \int_{-1}^{1} D(x(\xi))p_m(\xi)\frac{h_e}{2}d\xi, \\
x(\xi) &= \frac{1}{2}(h_e\xi + x_e + x_{e+1}), \quad h_e = x_{e+1} - x_e, \quad \text{(linear)} \\
x(\xi) &= \frac{1}{2}(h_e\xi + x_{2e-1} + x_{2e+1}), \quad h_e = x_{2e+1} - x_{2e-1} \quad \text{(quadratic).}
\end{aligned}$$

6. Element assembly:

   Linear shape functions:

$$\begin{aligned}
K &= \sum_{e=1}^{N} K^e \\
&= \begin{pmatrix}
k^1_{11} & k^1_{12} & 0 & 0 & \cdots & 0 & 0 & 0 \\
k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} & 0 & \cdots & 0 & 0 & 0 \\
0 & k^2_{21} & k^2_{22} + k^3_{11} & k^3_{12} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & k^{N-1}_{21} & k^{N-1}_{22} + k^N_{11} & k^N_{12} \\
0 & 0 & 0 & 0 & \cdots & 0 & k^N_{21} & k^N_{22}
\end{pmatrix}, \\
F &= \begin{pmatrix}
f^1_1 \\
f^1_2 + f^2_1 \\
f^2_2 + f^3_1 \\
\vdots \\
f^{N-1}_2 + f^N_1 \\
f^N_2
\end{pmatrix}.
\end{aligned}$$

Quadratic shape functions:

$$
\begin{aligned}
K &= \sum_{e=1}^{N} K^e \\
&= \begin{pmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{12}^2 & k_{13}^2 & \cdots & 0 & 0 & 0 \\
0 & 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 & \cdots & 0 & 0 & 0 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & k_{33}^{N-1} + k_{11}^N & k_{12}^N & k_{13}^N \\
0 & 0 & 0 & 0 & 0 & \cdots & k_{21}^N & k_{22}^N & k_{23}^N \\
0 & 0 & 0 & 0 & 0 & \cdots & k_{31}^N & k_{32}^N & k_{33}^N
\end{pmatrix}, \\
F &= \begin{pmatrix}
f_1^1 \\
f_2^1 \\
f_3^1 + f_1^2 \\
f_2^2 \\
f_3^2 + f_1^3 \\
\vdots \\
f_3^{N-1} + f_1^N \\
f_2^N \\
f_3^N
\end{pmatrix}.
\end{aligned}
$$

7. Boundary condition treatment. For the Dirichlet boundary conditions: $y(0) = y_0, \quad y(L) = y_L$, blast the diagonals

$$
\begin{aligned}
&\begin{pmatrix}
B & k_{12}^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & k_{21}^{N-1} & k_{22}^{N-1} + k_{11}^N & k_{12}^N \\
0 & 0 & 0 & 0 & \cdots & 0 & k_{21}^N & B
\end{pmatrix}
\begin{pmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \\ y_{N+1}
\end{pmatrix} \\
&= \begin{pmatrix}
y_0 \times B \\
f_2^1 + f_1^2 \\
f_2^2 + f_1^3 \\
\vdots \\
f_2^{N-1} + f_1^N \\
y_L \times B
\end{pmatrix},
\end{aligned}
$$

For the Neumann boundary conditions: $A(0)y'(0) = -q_0$, $A(L)y'(L) = q_L$, if $C(x) = 0$, then specify the value of one of y_i $(i = 1, 2, \cdots, N+1)$.

8. Gaussian quadrature:
$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} W_i f(x_i).$$
9. Physical FEM and mathematical FEM

- **You should be able to**

1. derive the weak form of an ODE;
2. use Galerkin's approximation in the weak form;
3. derive the linear and other high order shape functions;
4. sketch the linear and other high order basis functions;
5. calculate element matrices;
6. assemble element matrices;
7. apply boundary conditions;
8. use Gaussian quadrature to calculate numerical integrals;
9. generate a mesh;
10. use and modify the code ode2.m to solve one dimensional problems numerically;
11. specify the boundary conditions and coefficients in the code ode2.m;
12. analyze your numerical results;
13. determine when to use the physical FEM and mathematical FEM.

- **What "why questions" should you be able to answer?**

1. Why do we need the weak form of an ODE?
2. Why do we need to take the test functions to be the basis functions ϕ_i $(i = 1, 2, \cdots, N+1)$ in the weak form?
3. Why do we keep the known boundary node values in the Galerkin's approximations?
4. Why do we need to blast the diagonals corresponding the known node values?
5. Why do we need to specify one of the node values if $C(x) = 0$ in the case of Neumann boundary conditions?
6. Why do we need to introduce the Master element $[-1, 1]$ and the transformation from a general element to the Master one?

7. Why do we need higher order shape functions but not too high?
8. Why is a basis function required to be equal 1 at one node and zero at all other nodes?
9. Why are the functions defined by shape functions element by element are basis functions? are they linearly independent? why?
10. Why are the Galerkin's approximations working?

- **What possible extensions should you continue to pursue?**

1. What if the distributed source $D(x)$ is a concentrated one, the Dirac delta $\delta(x-x_0)$ at a point x_0?
2. What if the wire is composed of two or more different materials?
3. General natural boundary condition: $A(0)y'(0) + \alpha_0 y(0) = -q_0$, $A(L)y'(L) + \alpha_L y(L) = q_L$.
4. Other approximation methods like the Ritz method and Least Squares.

Question. Consider the boundary value problem

$$\begin{aligned} xy'' + y &= 1, \quad 1 < x < 2, \\ y(1) &= 2, \quad y'(2) = 3. \end{aligned}$$

1. Determine the coefficients $A(x), B(x), C(x), D(x)$ and the flux q_L at the right end $x = 2$.

2. Derive the weak form.

3. Specify the boundary conditions in the following MESH data file

```
%  MESH data file
%----------------------------------------------
% NUMNP  + 3 dummy numbers
%----------------------------------------------
   4       0      0      0
%----------------------------------------------
% XORD  NPBC     Y      Q
%----------------------------------------------
   1
   1.2     0      0      0
   1.6     0      0      0
   2
```

4. For a mesh of four elements of the same length and linear shape functions, calculate the entry k_{12}^2 by using the master element (answer: -28/27).