AM466/562: Finite Element Method

Solution of Bonus Problem

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Consider the boundary value problem

$$\begin{aligned} -y''(x) &= f(x), \quad 0 < x < 1, \\ y(0) &= y(1) = 0, \end{aligned}$$

where f is any given function. The weak form of the problem is

$$\int_0^1 y'(x)u'(x)dx = \int_0^1 f(x)u(x)dx, \quad \text{for all } u \in H_0^1(0,1). \tag{1}$$

Let $x_1 = 0 < x_2 < \cdots < x_N < x_{N+1} = 1$ be the mesh of $[0,1]$ and $\phi_1(x), \phi_2(x), \cdots, \phi_N(x)$, $\phi_{N+1}(x)$ be the linear finite element basis functions, and

$$H^{N-2} = \{a_2\phi_2(x) + \cdots + a_N\phi_N(x) \mid a_2, \cdots, a_N \text{ any real numbers } \} \subset H_0^1(0,1).$$

Let y_h be the linear finite element approximation of y in the finite dimensional space H^{N-2} that satisfies

$$\int_0^1 y_h'(x)u'(x)dx = \int_0^1 f(x)u(x)dx, \quad \text{for all } u \in H^{N-2}. \tag{2}$$

You might have noticed that the linear finite element approximation y_h is exactly equal to y at the node points x_i, $i = 1, 2, \cdots, N, N+1$. This problem is to ask you to prove this fact.

(i) (5 points) For any $i = 2, \cdots, N$, construct a function $G_i(x) \in H^{N-2}$ such that

$$\int_0^1 u'(x)G_i'(x)dx = u(x_i), \quad \text{for all } u \in H_0^1(0,1). \tag{3}$$

Solution 1. Since $G_i(x) \in H^{N-2}$, $G_i(0)$ and $G_i(1)$ must be equal to zero and we can also guess that $G_i(x)$ should be linear on $[0, x_i]$ and $[x_i, 1]$. By condition (3), we can guess that $G_i(x)$ is not differentiable at x_i. We then conclude that $G_i(x)$ should have the following form

$$G_i(x) = \begin{cases} ax, & 0 \le x \le x_i, \\ \frac{ax_i(1-x)}{1-x_i}, & x_i \le x \le 1, \end{cases}$$

where the constant a is to be determined. To determine a, we substitute it into (3) and derive by noting that $u(0) = u(1) = 0$ that

$$\begin{aligned} u(x_i) &= \int_0^1 u'(x)G_i'(x)dx \\ &= \int_0^{x_i} u'(x)G_i'(x)dx + \int_{x_i}^1 u'(x)G_i'(x)dx \\ &= a\int_0^{x_i} u'(x)dx - \frac{ax_i}{1-x_i}\int_{x_i}^1 u'(x)dx \\ &= au(x_i) + \frac{ax_i}{1-x_i}u(x_i) \\ &= \frac{a}{1-x_i}u(x_i). \end{aligned}$$

So $a = 1 - x_i$ and then

$$G_i(x) = \begin{cases} (1-x_i)x, & 0 \le x \le x_i, \\ x_i(1-x), & x_i \le x \le 1, \end{cases}$$

Solution 2. For any $0 < t < 1$, we define

$$u_t(x) = \begin{cases} x, & 0 \le x \le t, \\ \frac{t(1-x)}{1-t}, & t \le x \le 1. \end{cases}$$

It is clear that $u_t \in H_0^1(0,1)$. Since $G_i(0) = G_i(1) = 0$, it therefore follows from (3) that

$$\begin{aligned} u_t(x_i) &= \int_0^1 u_t'(x)G_i'(x)dx \\ &= \int_0^t u_t'(x)G_i'(x)dx + \int_t^1 u_t'(x)G_i'(x)dx \\ &= \int_0^t G_i'(x)dx - \frac{t}{1-t}\int_t^1 G_i'(x)dx \\ &= G_i(t) + \frac{t}{1-t}G_i(t) \\ &= \frac{1}{1-t}G_i(t). \end{aligned}$$

We then derive that

$$\begin{aligned} G_i(t) &= (1-t)u_t(x_i) \\ &= \begin{cases} (1-x_i)t, & 0 \le t \le x_i, \\ x_i(1-t), & x_i \le t \le 1. \end{cases} \end{aligned}$$

One can readily verify that such $G_i \in H^{N-2}$ and it satisfies (3). Indeed, we have for all $u \in H_0^1(0,1)$

$$\begin{aligned} \int_0^1 u'(x)G_i'(x)dx &= \int_0^{x_i} u'(x)G_i'(x)dx + \int_{x_i}^1 u'(x)G_i'(x)dx \\ &= (1-x_i)\int_0^{x_i} u'(x)dx - x_i\int_{x_i}^1 u'(x)dx \\ &= u(x_i)(1-x_i) + x_i u(x_i) \\ &= u(x_i). \end{aligned}$$

(ii) (5 points) Show that $y_h(x_i) = y(x_i)$, $i = 2, \cdots, N$.

Solution. Since $u = y - y_h \in H_0^1(0,1)$ and $G_i \in H^{N-2}$, it follows from (1), (2), and (3) that

$$\begin{aligned} y(x_i) - y_h(x_i) &= \int_0^1 (y'(x) - y_h'(x))G_i'(x)dx \\ &= \int_0^1 f(x)G_i'(x)dx - \int_0^1 f(x)G_i'(x)dx \\ &= 0. \end{aligned}$$

So $y_h(x_i) = y(x_i)$.