Solutions of the Problem in the Review of 2D Problems

Self-test problem (Do the problem first before looking at the solutions). Consider the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega = (0,1) \times (0,1),$$

subject to the boundary conditions

$$\begin{aligned} u(1,y) &= 1, \quad \frac{\partial u}{\partial x}(0,y) = 0, \quad 0 \le y \le 1, \\ \frac{\partial u}{\partial x}(x,0) &= 0, \quad \frac{\partial u}{\partial x}(x,1) = 1 - u(x,1), \quad 0 \le x \le 1. \end{aligned}$$

1. Derive the weak form.

   Multiplying the equation by a test function $v \in H^1(\Omega)$ with $v(1,y) = 0$ and integrating over Ω give

   $$\int_\Omega \left(\frac{\partial^2 u}{\partial x^2} v + \frac{\partial^2 u}{\partial y^2} v \right) dxdy = 0.$$

   Integration by parts gives

   $$\int_0^1 \frac{\partial u}{\partial x} v\Big|_{x=0}^1 dy + \int_0^1 \frac{\partial u}{\partial y} v\Big|_{y=0}^1 dx - \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dxdy = 0.$$

   Applying the boundary conditions, we obtain the weak form

   $$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dxdy + \int_0^1 u(x,1)v(x,1)dx = \int_0^1 v(x,1)dx,$$
   $$\text{any } v \in H^1(\Omega) \text{ with } v(1,y) = 0.$$

2. For the mesh of one rectangular element of the unit square, write out the connectivity matrix for the mesh.

   $$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

3. Write out four shape functions.

$$
\begin{aligned}
N_1(x,y) &= (1-x)(1-y),\\
N_2(x,y) &= x(1-y),\\
N_3(x,y) &= xy,\\
N_1(x,y) &= (1-x)y.
\end{aligned}
$$

4. Compute the contribution to the stiffness matrix from the domain.

$$
K^1_{ij} = \int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy, \quad i,j = 1,2,3,4.
$$

$$
\begin{aligned}
K^1_{11} &= \int_\Omega \left((1-y)^2 + (1-x)^2\right) dxdy = \frac{2}{3},\\
K^1_{12} &= K^1_{21} = \int_\Omega \left(-(1-y)^2 + (1-x)x\right) dxdy = -\frac{1}{6},\\
K^1_{13} &= K^1_{31} = \int_\Omega \left(-(1-y)y - (1-x)x\right) dxdy = -\frac{1}{3},\\
K^1_{14} &= K^1_{41} = \int_\Omega \left((1-y)y - (1-x)^2\right) dxdy = -\frac{1}{6},\\
K^1_{22} &= \int_\Omega \left((1-y)^2 + x^2\right) dxdy = \frac{2}{3},\\
K^1_{23} &= K^1_{32} = \int_\Omega \left((1-y)y - x^2\right) dxdy = -\frac{1}{6},\\
K^1_{24} &= K^1_{42} = \int_\Omega \left(-(1-y)y - x(1-x)\right) dxdy = -\frac{1}{3},\\
K^1_{33} &= \int_\Omega \left(y^2 + x^2\right) dxdy = \frac{2}{3},\\
K^1_{34} &= K^1_{43} = \int_\Omega \left(-y^2 + x(1-x)\right) dxdy = -\frac{1}{6},\\
K^1_{44} &= K^1_{32} = \int_\Omega \left(y^2 + x^2\right) dxdy = \frac{2}{3}.
\end{aligned}
$$

So

$$K^1 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

5. Compute the contribution to the stiffness matrix from the boundary.

$$\begin{aligned}
\Sigma^1_{K,33} &= \int_0^1 N_3(x,1)N_3(x,1)dx = \int_0^1 x^2 dx = \frac{1}{3}, \\
\Sigma^1_{K,34} &= \Sigma^1_{K,43} = \int_0^1 N_3(x,1)N_4(x,1)dx = \int_0^1 x(1-x)dx = \frac{1}{6}, \\
\Sigma^1_{K,44} &= \int_0^1 N_4(x,1)N_4(x,1)dx = \int_0^1 (1-x)^2 dx = \frac{1}{3}.
\end{aligned}$$

Since other $\Sigma^1_{K,ij} = 0$, we have

$$\Sigma^1_K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

6. Compute the force vector from the boundary.

$$\begin{aligned}
\Sigma^1_{q,3} &= \int_0^1 N_3(x,1)dx = \int_0^1 x dx = \frac{1}{2}, \\
\Sigma^1_{q,4} &= \int_0^1 N_4(x,1)dx = \int_0^1 (1-x)dx = \frac{1}{2}.
\end{aligned}$$

Since other $\Sigma^1_{q,i} = 0$, we have

$$\Sigma^1_q = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

7. Assemble these matrices and write out the linear system for the unknown nodal values.

$$K = K^1 + \Sigma_K^1 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & 1 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & 1 \end{pmatrix}.$$

The linear system for the unknown nodal values is

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & 1 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

8. solve the system to obtain nodal values.

Since $u_2 = u_3 = 1$ are known, the above system is reduced to

$$\begin{aligned} \frac{2}{3}u_1 - \frac{1}{6} - \frac{1}{3} - \frac{1}{6}u_4 &= 0, \\ -\frac{1}{6}u_1 - \frac{1}{3} + u_4 &= \frac{1}{2}. \end{aligned}$$

Solving it gives $u_1 = u_4 = 1$.