Stabilization and Controllability for the Transmission Wave Equation

Weijiu Liu

***Abstract*—In this paper, we address the problem of control of the transmission wave equation. In particular, we consider the case where, due to total internal reflection of waves at the interface, the system may not be controlled from exterior boundaries. We show that such a system can be controlled by introducing both boundary control along the exterior boundary and distributed control near the transmission boundary and give a physical explanation why the additional control near the transmission boundary might be needed for some domains.**

***Index Terms*—Stabilization, transmission wave equation.**

I. Introduction

THE aim of this paper is to address the problem of control of the transmission wave equation. More precisely we consider the case where, due to total internal reflection of waves at the interface, the system may not be controlled from exterior boundaries. This complements previous results by J. L. Lions [7] among others.

Let Ω be a bounded domain (open, nonempty, and connected) in $\mathbb{R}^n (n \geq 1)$ with a suitably smooth boundary $\Gamma = \partial\Omega$ which consists of two parts, Γ_1 and Γ_2 (see Fig. 1) Let Γ_{tr} be a smooth hypersurface, which separates Ω into two domains Ω_1 and Ω_2. The following figure is typical domain of this kind.

Consider the problem of transmission of the wave equation

$$u_i'' - a_i \Delta u_i = 0 \quad \text{in } \Omega_i \times (0, \infty), \tag{1.1}$$

$$u_i = 0 \quad \text{on } (\Gamma \cap \partial\Omega_i) \times (0, \infty), \tag{1.2}$$

$$u_1 = u_2, \ \ a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \quad \text{on } \Gamma_{tr} \times (0, \infty), \tag{1.3}$$

$$u_i(0) = u_i^0, \ \ u_i'(0) = u_i^1 \quad \text{in } \Omega_i, \ i = 1, 2. \tag{1.4}$$

In the above problem, $u_i = u_i(x, t)$, a_1 and a_2 are positive constants, the prime $'$ denotes the derivate with respect to the time variable, Δ denotes the Laplace operator in the space variables and ν denotes the unit normal on Γ and Γ_{tr} directing toward the exterior of Ω and Ω_1 and $u_i(0)$ and $u_i'(0)$ denote the functions $x \to u_i(x, 0)$ and $x \to u_i'(x, 0)$, respectively. This transmission problem describes the wave propagation from one medium into another different medium, for instance, from air into glass, and therefore it is of practical significance.

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The author was with the Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada. He is now with the Department of Mathematics, the University of Cincinnati, Cincinnati, OH 45221–0025 USA (e-mail: weiliu@math.uc.edu).

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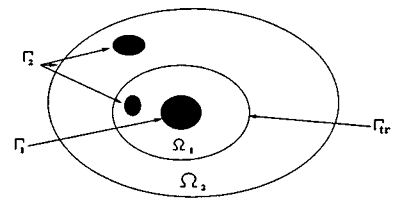

Fig. 1. A transmission domain.

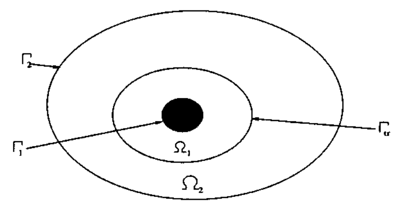

Fig. 2. A transmission domain.

In [7], Lions considered the problem of exact controllability for (1.1)–(1.4) with a domain as shown in Fig. 2 and established the results of exact controllability (see [7, p. 379, Th. 5.1] if

$$a_1 \geq a_2 \tag{1.5}$$

leaving the case where $a_1 < a_2$ as an open problem (see [7, p. 394, Prob. 8.1]). Similar condition was also imposed by Nicaise for an exact controllability problem (see [13, p. 1519, (H3)] and [14, p. 587, Th. 2.2]) and by Lagnese for problems of transmission of a class of second-order hyperbolic systems (see [4, p. 345, (1.14)]). In addition, the author and Williams considered the following problem of stabilization with a domain as shown in Fig. 2

$$u_i'' - a_i \Delta u_i = 0 \quad \text{in } \Omega_i \times (0, \infty), \tag{1.6}$$

$$u_1 = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.7}$$

$$\frac{\partial u_2}{\partial \nu} = -k u_2' \quad \text{on } \Gamma_2 \times (0, \infty), \tag{1.8}$$

$$u_1 = u_2, \ a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \quad \text{on } \Gamma_{tr} \times (0, \infty), \tag{1.9}$$

$$u_i(0) = u_i^0, \ u_i'(0) = u_i^1 \quad \text{in } \Omega_i, \ i = 1, 2, \tag{1.10}$$

and obtained the exponential stabilization under condition (1.5) (see [11, Th. 1.1]), leaving the case where $a_1 < a_2$ as an open problem again, where k is a positive constant. Therefore, the case where $a_1 < a_2$ has long become an interesting problem.

From the point of physical view, condition (1.5) is necessary if the control is applied only on the exterior boundary. To see this, we consider waves passing from a medium in which the speed is a_1 into a medium in which the speed a_2 is greater than

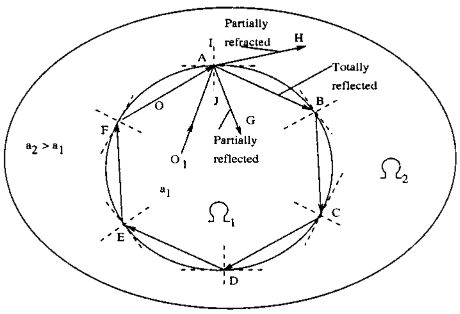

Fig. 3. Total internal reflection.

a_1 (see Fig. 3). If θ_1 is the angle of incidence (i.e., the angle O_1AJ) of a wave from O_1 and θ_2 the angle of refraction (i.e., the angle IAH), then, by the law of refraction (see, e.g., [17, p. 596]), we have

$$\sin\theta_2 = \frac{a_2}{a_1}\sin\theta_1. \tag{1.11}$$

Since $a_2 > a_1$, we can obtain the critical angle θ_c of incidence given by

$$\sin\theta_c = \frac{a_1}{a_2}.$$

When the angle of incidence is greater than the critical angle θ_c, the law of refraction (1.11) cannot be satisfied and there is no refracted wave in the second medium Ω_2. All the energy is reflected. This phenomenon is called *total internal reflection* because in optics the incident light is usually inside glass and reflected from the glass-air surface. This situation is illustrated in Fig. 3. The wave from the point O is totally reflected at the point A and then at the points B, C, D, E, F and finally totally back to the point O. In this way, the wave will propagate forever and never diminish. Therefore any control applied on the exterior boundary Γ of Ω can do nothing on such a wave. Consequently, in the case where $a_2 > a_1$, an additional control near the transmission boundary Γ_{tr} might be needed for some domains such as Fig. 3. However, if a domain Ω is like the one as shown in Fig. 4, the totally reflected waves will be absorbed or controlled when they reached the exterior boundary Γ of Ω and therefore there is no need to introduce additional control near the transmission boundary Γ_{tr}. For this reason, we shall consider only the domains as shown in Fig. 3.

In the rest of the paper, we consider the problem of stabilization for (1.1)–(1.4) in Section II. By introducing both boundary feedback control and distributed feedback control near the transmission boundary Γ_{tr}, we show that the controlled system is exponentially stable without any restriction on a_1 and a_2 and the transmission boundary Γ_{tr}. In Section III, we discuss the problem of exact controllability and prove that problem (1.1)–(1.4) is exactly controllable under the same assumptions.

We note that our paper has not solved an open problem raised by Lions in [7, p. 394, Prob. 8.1] since we introduce the additional distributed feedback control near the transmission boundary Γ_{tr} and this makes the problem much simpler than the original open problem in which only the boundary control is allowed. However, the phenomenon of total internal reflection shows that the original open problem for some domains such as Fig. 3 might have no solutions if the control is allowed only on the exterior boundary.

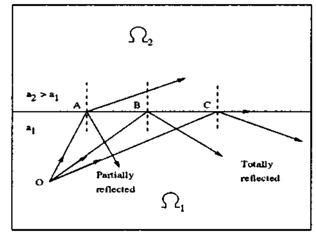

Fig. 4. Total internal reflection.

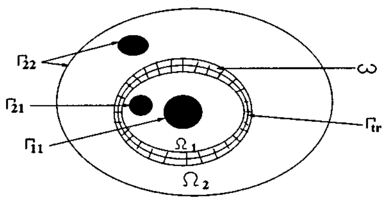

Fig. 5. A transmission domain.

II. Stabilization

Let Ω be a bounded domain (open, nonempty, and connected) in $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\Gamma = \partial\Omega$ which consists of two parts, Γ_1 and Γ_2. Γ_1 is assumed to be either empty or to have a nonempty interior and $\Gamma_2 \neq \emptyset$ and relatively open in Γ. Let Γ_{tr} be a smooth hypersurface, which separates Ω into two domains Ω_1 and Ω_2. Let ω be a domain near Γ_{tr} such that $\Gamma_{tr} \subset \omega \subset \Omega$. Assume

$$\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset, \quad \overline{\Gamma}_1 \cap \overline{\Gamma}_{tr} = \emptyset, \quad \overline{\Gamma}_{tr} \cap \overline{\Gamma}_2 = \emptyset.$$

We denote

$$\omega_i = \Omega_i \cap \omega, \quad \Gamma_{ij} = \Gamma_i \cap \partial\Omega_j, \quad i, j = 1, 2.$$

Such a domain is illustrated in Fig. 5.

In addition to boundary feedback control (1.8), we introduce a localizedly distributed feedback control near the transmission boundary Γ_{tr} as follows

$$u_i'' - a_i\Delta u_i + b\chi_{\omega_i}u_i' = 0 \qquad \text{in } \Omega_i \times (0,\infty), \tag{2.1}$$

$$u_i = 0 \qquad \text{on } \Gamma_{1i} \times (0,\infty), \tag{2.2}$$

$$\frac{\partial u_i}{\partial \nu} = -ku_i' \qquad \text{on } \Gamma_{2i} \times (0,\infty), \tag{2.3}$$

$$u_1 = u_2, \;\; a_1\frac{\partial u_1}{\partial \nu} = a_2\frac{\partial u_2}{\partial \nu} \qquad \text{on } \Gamma_{tr} \times (0,\infty), \tag{2.4}$$

$$u_i(0) = u_i^0, \;\; u_i'(0) = u_i^1 \qquad \text{in } \Omega_i,\; i = 1, 2, \tag{2.5}$$

where b and k are positive constants and χ_{ω_i} denotes the characteristic function of ω_i in Ω_i.

To state our results, we introduce notations used throughout the paper. For a domain O in $\mathbb{R}^n$, we denote by $H^s(O)$ the usual Sobolev space for any $s \in \mathbb{R}$ (see, e.g., [1]). If $\Gamma_1 = \emptyset$, we define

$$\begin{aligned} V_{\Gamma_1} = \Bigg\{ &\left(u_1^0, u_2^0, u_1^1, u_2^1\right) \in H^1(\Omega_1) \times H^1(\Omega_2) \\ &\times L^2(\Omega_1) \times L^2(\Omega_2) \colon\; u_1^0 = u_2^0 \text{ on } \Gamma_{tr} \text{ and} \\ &\times \sum_{i=1}^{2}\left(\int_{\Gamma_{2i}} ka_iu_i^0\,d\Gamma + b\int_{\omega_i} u_i^0\,dx + \int_{\Omega_i} u_i^1\,dx\right) = 0 \Bigg\}. \end{aligned} \tag{2.6}$$

If $\Gamma_1 \neq \emptyset$, we define

$$V_{\Gamma_1} = \{(u_1^0, u_2^0, u_1^1, u_2^1) \in H^1(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_1) \times L^2(\Omega_2)\colon\ u_1^0 = u_2^0 \text{ on } \Gamma_{tr} \text{ and } u_i^0 = 0 \text{ on } \Gamma_{1i},\ i = 1, 2\}. \tag{2.7}$$

As in [9], we can readily prove that there exists a positive constant C such that

$$\sum_{i=1}^{2} \int_{\Omega_i} \left|u_i^0\right|^2 dx \geq C \sum_{i=1}^{2} \int_{\Omega_i} \left(|u_i^1|^2 + a_i|\nabla u_i^0|^2\right) dx, \quad \forall (u_1^0, u_2^0, u_1^1, u_2^1) \in V_{\Gamma_1}. \tag{2.8}$$

Hence, the norm on V_{Γ_1}

$$\left\|(u_1^0, u_2^0, u_1^1, u_2^1)\right\|_{V_{\Gamma_1}} = \left(\frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_i} \left(\left|u_i^1\right|^2 + a_i\left|\nabla u_i^0\right|^2\right) dx\right)^{1/2} \tag{2.9}$$

is equivalent to the usual one induced by $H^1(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_1) \times L^2(\Omega_2)$. As usual, we can easily show that system (2.1)–(2.5) generates a strongly continuous semigroup on V_{Γ_1}.

The energy of system (2.1)–(2.5) is defined by

$$E(t) = \|(u_1, u_2, u_1', u_2')\|^2_{V_{\Gamma_1}} = \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_i} \left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right) dx. \tag{2.10}$$

We can readily verify that

$$E'(t) = -k \sum_{i=1}^{2} \int_{\Gamma_{2i}} a_i|u_i'|^2\, d\Gamma - \sum_{i=1}^{2} \int_{\omega_i} b|u_i'|^2\, dx. \tag{2.11}$$

Hence the energy decreases with time. Indeed, we have the following exponential stability theorem.

Theorem 2.1: Let ν denote the unit normal on Γ and Γ_{tr} pointing toward the exterior of Ω and Ω_1. Suppose that there exists an $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \leq 0 \qquad \text{on } \Gamma_1 \tag{2.12}$$

where $m = m(x) = x - x_0$. Then there are positive constants M, τ such that

$$E(t) \leq Me^{-\tau t}E(0) \qquad \text{for } t \geq 0 \tag{2.13}$$

for all solutions of (2.1)–(2.5) with $(u_1^0, u_2^0, u_1^1, u_2^1) \in V_{\Gamma_1}$.

Obviously, if the domain Ω has only one hole, then the boundary control can be supported only on the exterior part of the boundary of Ω by taking an x_0 inside the hole. However, if the domain Ω has more than one holes, except for one of them, the boundary control has to be applied to the boundaries of all other holes.

Remark 2.1: In Theorem 2.1, no geometric conditions are imposed on Ω. Such an improvement without geometric conditions was due to Lasiecka and Triggiani [5] and here we simply follow their idea and then nothing is new in this aspect. We note that such an improvement is established at the cost that the support Γ_2 of boundary control may contain points satisfying $m \cdot \nu \leq 0$ and therefore bigger than the usual set

$$\Gamma(x_0) = \{x \in \Gamma\colon\ m(x) \cdot \nu(x) > 0\}. \tag{2.14}$$

If we want to reduce the support Γ_2 to $\Gamma(x_0)$, we may lose something again, that is, the solution of (2.1)–(2.5) may have singularity at points $x \in \overline{\Gamma^*(x_0)} \cap \overline{\Gamma(x_0)}$ since, in general, $\overline{\Gamma^*(x_0)} \cap \overline{\Gamma(x_0)} \neq \emptyset$, where $\Gamma^*(x_0) = \Gamma - \Gamma(x_0)$ (see Fig. 6). For the discussion of such a case, we refer to [3].

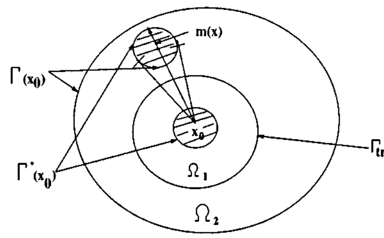

Fig. 6. Boundary partitions $\Gamma(x_0)$ and $\Gamma^*(x_0)$.

Remark 2.2: Once we obtained the exponential stabilization, the exact controllability for (1.1)–(1.4) can be readily established by applying Russell's "controllability via stabilizability" principle (see, e.g., [15]) as we did in [11].

We now prove Theorem 2.1. The idea of the proof is simple. It suffices to show that there exist positive constants $T > 0$ and $0 < \rho < 1$ such that (see, e.g., [2], [5], and [10])

$$E(t) \leq \rho E(0), \qquad \forall\, t \geq T. \tag{2.15}$$

However, the verification of this inequality is not easy. For this, we employ the classical multiplier method originated by Lax, Morawetz, Phillips, Ralston, and Strauss (see [6], [12], and [16]).

Proof of Theorem 2.1: For $T > 0$, integrating (2.11) from 0 to T, we obtain

$$E(T) + k \sum_{i=1}^{2} \int_0^T \int_{\Gamma_{2i}} a_i|u_i'|^2\, d\Gamma\, dt + \sum_{i=1}^{2} \int_0^T \int_{\omega_i} b|u_i'|^2\, dx\, dt = E(0). \tag{2.16}$$

If we can prove that there exist a sufficiently large $T > 0$ and a corresponding constant $K(T) > 0$ such that

$$K(T)E(T) \leq k \sum_{i=1}^{2} \int_0^T \int_{\Gamma_{2i}} a_i|u_i'|^2\, d\Gamma\, dt + \sum_{i=1}^{2} \int_0^T \int_{\omega_i} b|u_i'|^2\, dx\, dt \tag{2.17}$$

then (2.15) can be established. Therefore, our proof is reduced to prove (2.17).

For any $0 < \delta < T$, we denote

$$\begin{aligned} Q_i &= \Omega_i \times (\delta, T - \delta) \\ Q_{\omega_i} &= \omega_i \times (\delta, T - \delta) \\ \Sigma_i &= \partial\Omega_i \times (\delta, T - \delta) \end{aligned} \tag{2.18}$$

$$\begin{aligned} \Sigma_{ij} &= \Gamma_{ij} \times (\delta, T - \delta) \\ \Sigma_{tr} &= \Gamma_{tr} \times (\delta, T - \delta), \qquad i = 1, 2. \end{aligned} \tag{2.19}$$

Since we assume that the unit normal ν on Γ_{tr} points toward the exterior of Ω_1, we have (see Fig. 5)

$$\Sigma_1 = \Sigma_{11} \cup \Sigma_{21} \cup \Sigma_{tr} \tag{2.20}$$

$$\Sigma_2 = \Sigma_{12} \cup \Sigma_{22} \cup (-\Sigma_{tr}) \tag{2.21}$$

where $-\Sigma_{tr}$ means that the unit normal ν on Γ_{tr} points toward the interior of Ω_2.

Let $l(x) = (l_1(x), \ldots, l_n(x))$ be a vector field of class $C^2(\overline{\Omega})$. Multiplying equation (2.1) by $l_k(\partial u_i/\partial x_k)$ $(i = 1, 2)$ and integrating on Q_i by parts, we have

$$\begin{aligned}0 = &\left(u_i'(t),\, l_k \frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta} - \frac{1}{2}\int_{\Sigma_i} l_k\nu_k|u_i'|^2\, d\Sigma \\ &+ \frac{1}{2}\int_{Q_i} \operatorname{div}(l)|u_i'|^2\, dx\, dt - a_i\int_{\Sigma_i} \frac{\partial u_i}{\partial \nu}\, l_k\, \frac{\partial u_i}{\partial x_k}\, d\Sigma \\ &+ a_i\int_{Q_i} \frac{\partial u_i}{\partial x_j}\,\frac{\partial l_k}{\partial x_j}\,\frac{\partial u_i}{\partial x_k}\, dx\, dt + \frac{a_i}{2}\int_{\Sigma_i} l_k\nu_k|\nabla u_i|^2\, d\Sigma \\ &- \frac{a_i}{2}\int_{Q_i} \operatorname{div}(l)|\nabla u_i|^2\, dx\, dt + \int_{Q_i} b\chi_{\omega_i} l_k\, \frac{\partial u_i}{\partial x_k}\, u_i'\, dx\, dt.\end{aligned} \tag{2.22}$$

Multiplying (2.1) by u_i $(i = 1, 2)$ and integrating on Q_i by parts, we obtain

$$\begin{aligned}\sum_{i=1}^2 &\int_{Q_i} \left(|u_i'|^2 - a_i|\nabla u_i|^2\right) dx\, dt \\ &= \sum_{i=1}^2 (u_i'(t), u_i(t))|_\delta^{T-\delta} + \sum_{i=1}^2 \int_{Q_i} b\chi_{\omega_i} u_i' u_i\, dx\, dt \\ &\quad - \sum_{i=1}^2 a_i\int_{\Sigma_{2i}} u_i\, \frac{\partial u_i}{\partial \nu}\, d\Sigma.\end{aligned} \tag{2.23}$$

Summing (2.22) for i with $l = m$, using (2.20), (2.21) and (2.23) and noting that $u_i = 0$ on Σ_{1i} and $u_1 = u_2$ on Σ_{tr}, we obtain

$$\begin{aligned}\int_\delta^{T-\delta} &E(t)\, dt \\ &= -\sum_{i=1}^2 \left(u_i'(t),\, \frac{n-1}{2}\, u_i(t) + m_k\, \frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta} \\ &\quad + \frac{(n-1)}{2} \sum_{i=1}^2 a_i\int_{\Sigma_{2i}} u_i\, \frac{\partial u_i}{\partial \nu}\, d\Sigma \\ &\quad - \sum_{i=1}^2 \int_{Q_i} b\chi_{\omega_i}\left(\frac{n-1}{2}\, u_i + m_k\, \frac{\partial u_i}{\partial x_k}\right) u_i'\, dx\, dt \\ &\quad + \tfrac{1}{2}\sum_{i=1}^2 \int_{\Sigma_{2i}} m\cdot\nu|u_i'|^2\, d\Sigma \\ &\quad + \int_{\Sigma_{tr}} \left(a_1\, \frac{\partial u_1}{\partial \nu}\, m_k\, \frac{\partial u_1}{\partial x_k} - a_2\, \frac{\partial u_2}{\partial \nu}\, m_k\, \frac{\partial u_2}{\partial x_k}\right. \\ &\qquad\qquad \left. + \frac{a_2}{2}\, m_k\nu_k|\nabla u_2|^2 - \frac{a_1}{2}\, m_k\nu_k|\nabla u_1|^2\right) d\Sigma \\ &\quad + \sum_{i=1}^2 a_i\int_{\Sigma_{1i}} \left(\frac{\partial u_i}{\partial \nu}\, m_k\, \frac{\partial u_i}{\partial x_k} - \frac{1}{2}\, m_k\nu_k|\nabla u_i|^2\right) d\Sigma \\ &\quad + \sum_{i=1}^2 a_i\int_{\Sigma_{2i}} \left(\frac{\partial u_i}{\partial \nu}\, m_k\, \frac{\partial u_i}{\partial x_k} - \frac{1}{2}\, m_k\nu_k|\nabla u_i|^2\right) d\Sigma.\end{aligned} \tag{2.24}$$

To estimate the right hand side of (2.24), only the transmission term needs a special care and all other terms can be handled in the usual way. However, for reader's convenience, we give detailed estimates for all terms. In what follows, the C denotes a generic positive constant $C(n, x^0, \Omega)$, *independent of* T, which may vary from line to line, while the $K(T)$ denoting a generic positive constant $K(n, x^0, \Omega, T)$, dependent of T.

- *First Term*. It follows from (2.8) and (2.11) that:

$$\left|\sum_{i=1}^2 \left(u_i'(t),\, \frac{n-1}{2}\, u_i(t) + m_k\, \frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta}\right| \le CE(\delta). \tag{2.25}$$

- *Second Term*. By Young's inequality and the trace theorem (see, e.g., [8, p. 39]), we obtain

$$\begin{aligned}\frac{n-1}{2} &\left|\sum_{i=1}^2 a_i\int_{\Sigma_{2i}} u_i\, \frac{\partial u_i}{\partial \nu}\, d\Sigma\right| \\ &\le C\sum_{i=1}^2 \int_\delta^{T-\delta} \|u_i(t)\|_{H^1(\Omega_i)} \left(\int_{\Gamma_{2i}} |u_i'(t)|^2\, d\Gamma\right)^{1/2} dt \\ &\le \tfrac{1}{4}\int_\delta^{T-\delta} E(t)\, dt + C\sum_{i=1}^2 \int_{\Sigma_{2i}} |u_i'|^2\, d\Sigma \qquad \text{(use (2.8))}.\end{aligned} \tag{2.26}$$

- *Third Term*. By (2.8) and Young's inequality it follows that:

$$\begin{aligned}&\left|\sum_{i=1}^2 \int_{Q_i} b\chi_{\omega_i}\left(\frac{n-1}{2}\, u_i + m_k\, \frac{\partial u_i}{\partial x_k}\right) u_i'\, dx\, dt\right| \\ &\qquad \le \tfrac{1}{4}\int_\delta^{T-\delta} E(t)\, dt + C\sum_{i=1}^2 \int_{Q_{\omega_i}} b|u_i'|^2\, dx\, dt.\end{aligned} \tag{2.27}$$

- *Fifth Term*. Let us choose the open subsets ω', ω'' and the vector field l to be such that

$$\begin{aligned}l = m \quad &\text{on } \Gamma_{tr},\ \operatorname{supp} l \subset \omega' \text{ and} \\ &\Gamma_{tr} \subset \omega' \subset \overline{\omega'} \subset \omega'' \subset \overline{\omega''} \subset \omega.\end{aligned} \tag{2.28}$$

Then $l = 0$ near Γ. It therefore follows from (2.22) that:

$$\begin{aligned}\int_{\Sigma_{tr}} &\left(a_1\, \frac{\partial u_1}{\partial \nu}\, m_k\, \frac{\partial u_1}{\partial x_k} - a_2\, \frac{\partial u_2}{\partial \nu}\, m_k\, \frac{\partial u_2}{\partial x_k}\right. \\ &\qquad \left. + \frac{a_2}{2}\, m_k\nu_k|\nabla u_2|^2 - \frac{a_1}{2}\, m_k\nu_k|\nabla u_1|^2\right) d\Sigma \\ &= \sum_{i=1}^2 \left(u_i'(t),\, l_k\, \frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta} \\ &\quad + \tfrac{1}{2}\sum_{i=1}^2 \int_{Q_i} \operatorname{div}(l)|u_i'|^2\, dx\, dt \\ &\quad + \sum_{i=1}^2 \int_{Q_i} a_i\, \frac{\partial u_i}{\partial x_j}\,\frac{\partial l_k}{\partial x_j}\,\frac{\partial u_i}{\partial x_k}\, dx\, dt \\ &\quad - \tfrac{1}{2}\sum_{i=1}^2 \int_{Q_i} a_i\operatorname{div}(l)|\nabla u_i|^2\, dx\, dt \\ &\quad + \sum_{i=1}^2 \int_{Q_i} b\chi_{\omega_i} l_k\, \frac{\partial u_i}{\partial x_k}\, u_i'\, dx\, dt \\ &\le CE(\delta) + C\sum_{i=1}^2 \int_{Q_{\omega_i'}} b|u_i'|^2\, dx\, dt \\ &\quad + C\sum_{i=1}^2 \int_{Q_{\omega_i'}} |\nabla u_i|^2\, dx\, dt\end{aligned} \tag{2.29}$$

where

$$Q_{\omega_i'} = \omega_i' \times (\delta, T-\delta), \qquad i = 1, 2.$$

Let $\theta = \theta(x)$ be a smooth function such that

$$\theta = 1 \quad \text{in } \omega' \text{ and } \theta = 0 \text{ in } \Omega - \overline{\omega''} \tag{2.30}$$

and

$$\max_{x \in \overline{\omega''}} \frac{|\nabla \theta|^2}{\theta} \le C. \tag{2.31}$$

Then $\theta = 0$ near Γ. For the existence of such a function, we refer to [7, p. 414]. Multiplying (2.1) by θu_i and integrating over Q_i by parts, we obtain

$$\begin{aligned}\sum_{i=1}^{2} a_i \int_{Q_i} \theta |\nabla u_i|^2 \, dx\, dt &= -\sum_{i=1}^{2} (u_i'(t), \theta u_i(t))\Big|_{\delta}^{T-\delta} + \sum_{i=1}^{2} \int_{Q_i} \theta |u_i'|^2 \, dx\, dt \\ &\quad - \sum_{i=1}^{2} \int_{Q_i} a_i u_i \nabla\theta \cdot \nabla u_i \, dx\, dt - \sum_{i=1}^{2} \int_{Q_i} b\chi_{\omega_i} \theta u_i u_i' \, dx\, dt \\ &\le CE(\delta) + C \sum_{i=1}^{2} \int_{Q_{\omega_i''}} b|u_i'|^2 \, dx\, dt \\ &\quad + \tfrac{1}{2} \sum_{i=1}^{2} a_i \int_{Q_i} \theta |\nabla u_i|^2 \, dx\, dt + C \sum_{i=1}^{2} \int_{Q_{\omega_i''}} |u_i|^2 \, dx\, dt\end{aligned} \tag{2.32}$$

where

$$Q_{\omega_i''} = \omega_i'' \times (\delta, T-\delta), \qquad i = 1, 2.$$

Hence, by (2.29), we deduce that

$$\begin{aligned}&\int_{\Sigma_{tr}} \left(a_1 \frac{\partial u_1}{\partial \nu} m_k \frac{\partial u_1}{\partial x_k} - a_2 \frac{\partial u_2}{\partial \nu} m_k \frac{\partial u_2}{\partial x_k} \right. \\ &\qquad \left. + \frac{a_2}{2} m_k \nu_k |\nabla u_2|^2 - \frac{a_1}{2} m_k \nu_k |\nabla u_1|^2 \right) d\Sigma \\ &\quad \le CE(\delta) + C \sum_{i=1}^{2} \int_{Q_{\omega_i}} b|u_i'|^2 \, dx\, dt \\ &\qquad + C \sum_{i=1}^{2} \int_{Q_{\omega_i}} |u_i|^2 \, dx\, dt.\end{aligned} \tag{2.33}$$

- *Sixth Term*. Since $u_i = 0$ on $\Gamma_{1i} \times (0, T)$, we deduce

$$\frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} - \frac{1}{2} m_k \nu_k |\nabla u_i|^2 = \frac{1}{2} m \cdot \nu \, \frac{\partial u_i}{\partial \nu}^2 \le 0. \tag{2.34}$$

- *Seventh Term*. To estimate the last term, we apply [5, Lemma 7.2, p. 218]. This lemma can be stated as follows: for arbitrarily small $\delta, \varepsilon > 0$ there exists a positive constant $K = K(T, \delta, \varepsilon)$ such that

$$\begin{aligned}&\sum_{i=1}^{2} \int_{\delta}^{T-\delta} \int_{\Gamma_{2i}} \left| \frac{\partial u_i}{\partial \tau} \right|^2 d\Gamma\, dt \\ &\quad \le K(T) \sum_{i=1}^{2} \left[\int_0^T \int_{\Gamma_{2i}} \left(\left| \frac{\partial u_i}{\partial \nu} \right|^2 + |u_i'|^2 \right) d\Gamma\, dt \right. \\ &\qquad \left. + \|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)} \right]\end{aligned} \tag{2.35}$$

where $\partial/\partial\tau$ denotes the tangential derivative on Γ_2 (for definition, see, e.g., [7, p. 137]). Although (2.35) was obtained for a second-order hyperbolic equation with smooth coefficients, it still holds for our transmission problem due to the local property of (2.35). Indeed, when we do the partition of unity for Γ_2 and flatten the boundary via a change of variable, we obtain a second-order hyperbolic equation with smooth coefficients in a half space of $\mathbb{R}^n$ since the coefficients near Γ_2 are constants. Moreover, the proof of Lemma 7.2 of [5, p. 218] works for the transmission problem simply because the elliptic estimate in [5, (7.10), p. 219] requires only *a priori* regularity of solution $u \in H^{1/2+\epsilon}(Q)$ and this is the regularity that the solution of our transmission problem has.[1] Noting that $|\nabla u_i|^2 = |\partial u_i/\partial\tau|^2 + |\partial u_i/\partial\nu|^2$ (see, e.g., [7, p. 137]), it therefore follows that:

$$\begin{aligned}&\left| \sum_{i=1}^{2} a_i \int_{\Sigma_{2i}} \left(\frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} - \frac{1}{2} m_k \nu_k |\nabla u_i|^2 \right) d\Sigma \right| \\ &\quad \le K(T) \sum_{i=1}^{2} \left[\int_0^T \int_{\Gamma_{2i}} \left(\left| \frac{\partial u_i}{\partial \nu} \right|^2 + |u_i'|^2 \right) d\Gamma\, dt \right. \\ &\qquad \left. + \|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)} \right].\end{aligned} \tag{2.36}$$

It therefore follows from (2.24)–(2.27), (2.33), (2.34) and (2.36) that:

$$\begin{aligned}\int_{\delta}^{T-\delta} E(t)\, dt &\le CE(\delta) + K(T) \sum_{i=1}^{2} \int_0^T \int_{\Gamma_{2i}} |u_i'|^2 \, d\Gamma\, dt \\ &\quad + C \sum_{i=1}^{2} \int_0^T \int_{\omega_i} b|u_i'|^2 \, dx\, dt \\ &\quad + K(T) \sum_{i=1}^{2} \|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)}.\end{aligned} \tag{2.37}$$

Since by (2.11)

$$\begin{aligned}E(\delta) &= E(T-\delta) + k \sum_{i=1}^{2} \int_{\delta}^{T-d} \int_{\Gamma_{2i}} a_i |u_i'|^2 \, d\Gamma\, dt \\ &\quad + b \sum_{i=1}^{2} \int_{\delta}^{T-d} \int_{\omega_i} |u_i'|^2 \, dx\, dt\end{aligned}$$

we obtain

$$\begin{aligned}&(T-2\delta)E(T-\delta) \\ &\quad \le \int_{\delta}^{T-\delta} E(t)\, dt \\ &\quad \le CE(T-\delta) + K(T) \sum_{i=1}^{2} \int_0^T \int_{\Gamma_{2i}} |u_i'|^2 \, d\Gamma\, dt \\ &\qquad + C \sum_{i=1}^{2} \int_0^T \int_{\omega_i} b|u_i'|^2 \, dx\, dt + K(T) \sum_{i=1}^{2} \|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)}\end{aligned} \tag{2.38}$$

[1]The author thanks Professor I. Lasiecka for pointing out this to him.

and then

$$\begin{aligned}(T-2\delta-C)E(T-\delta)&\\ \le K(T)\sum_{i=1}^{2}\int_0^T\int_{\Gamma_{2i}}|u_i'|^2\,d\Gamma\,dt&\\ +C\sum_{i=1}^{2}\int_0^T\int_{\omega_i}b|u_i'|^2\,dx\,dt+K(T)\sum_{i=1}^{2}\|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)}.&\end{aligned} \tag{2.39}$$

Using Lemma 2.1 below and by the usual compactness-uniqueness argument (see, e.g., [5] and [7, App. 1]), we can readily prove that

$$\begin{aligned}\sum_{i=1}^{2}\|u_i\|^2_{H^{1/2+\varepsilon}(Q_i)}\le K(T)\sum_{i=1}^{2}\int_0^T\int_{\Gamma_{2i}}|u_i'|^2\,d\Gamma\,dt&\\ +K(T)\sum_{i=1}^{2}\int_0^T\int_{\omega_i}b|u_i'|^2\,dx\,dt.&\end{aligned} \tag{2.40}$$

It therefore follows that:

$$\begin{aligned}(T-2\delta-C)E(T-\delta)\le K(T)\sum_{i=1}^{2}\int_0^T\int_{\Gamma_{2i}}|u_i'|^2\,d\Gamma\,dt&\\ +K(T)\sum_{i=1}^{2}\int_0^T\int_{\omega_i}b|u_i'|^2\,dx\,dt&\end{aligned} \tag{2.41}$$

which implies (2.17) for sufficiently large T. □

Lemma 2.1: If the solution of

$$u_i''-a_i\Delta u_i+b\chi_{\omega_i}u_i'=0 \quad \text{in } \Omega_i\times(0,T), \tag{2.42}$$
$$u_i=0 \quad \text{on } \Gamma_{1i}\times(0,T), \tag{2.43}$$
$$\frac{\partial u_i}{\partial\nu}=-ku_i' \quad \text{on } \Gamma_{2i}\times(0,T), \tag{2.44}$$
$$u_1=u_2,\ \ a_1\frac{\partial u_1}{\partial\nu}=a_2\frac{\partial u_2}{\partial\nu} \quad \text{on } \Gamma_{tr}\times(0,T), \tag{2.45}$$
$$u_i(0)=u_i^0,\ \ u_i'(0)=u_i^1 \quad \text{in } \Omega_i,\ i=1,2, \tag{2.46}$$

satisfies

$$u_i'=0 \quad \text{on } \Gamma_{2i}\times(0,T) \tag{2.47}$$

and

$$u_i'=0 \quad \text{in } \omega_i\times(0,T) \tag{2.48}$$

for sufficiently large $T>0$, then

$$u_i\equiv 0 \quad \text{in } \Omega_i\times(0,T). \tag{2.49}$$

Proof: Set $v_i=u_i'$. It is clear that v satisfies

$$v_i''-a_i\Delta v_i=0 \quad \text{in } \Omega_1\times(0,T), \tag{2.50}$$
$$v_i=0 \quad \text{on } \Gamma_i\times(0,T), \tag{2.51}$$
$$\frac{\partial v_i}{\partial\nu}=0 \quad \text{on } \Gamma_{2i}\times(0,T), \tag{2.52}$$
$$v_i=0 \quad \text{on } \omega_i\times(0,T). \tag{2.53}$$

Therefore, by [7, p. 92, Th. 8.2], we have

$$u_i'=v_i\equiv 0 \quad \text{in } \Omega_i\times(0,T). \tag{2.54}$$

It then follows that:

$$-a_i\Delta u_i=0 \quad \text{in } \Omega_1\times(0,T), \tag{2.55}$$
$$u_i=0 \quad \text{on } \Gamma_{1i}\times(0,T), \tag{2.56}$$
$$\frac{\partial u_i}{\partial\nu}=0 \quad \text{on } \Gamma_{2i}\times(0,T), \tag{2.57}$$
$$u_1=u_2,\ \ a_1\frac{\partial u_1}{\partial\nu}=a_2\frac{\partial u_2}{\partial\nu} \quad \text{on } \Gamma_{tr}\times(0,T), \tag{2.58}$$

which implies (2.49). □

III. Exact Controllability

In this section, the domains Ω, Ω_1 and Ω_2 are the same as in Section II, but the partition of the boundary Γ of Ω is different and we partition it as follows (see Fig. 6):

$$\Gamma(x_0)=\{x\in\Gamma:\ m(x)\cdot\nu(x)>0\} \tag{3.1}$$
$$\Gamma^*(x_0)=\Gamma-\Gamma(x_0) \tag{3.2}$$

where $x_0\in\mathbb{R}^n$ and $m=m(x)=x-x_0$. We denote

$$R(x_0)=\max_{x\in\Omega}|m| \tag{3.3}$$
$$\Gamma_i(x_0)=\Gamma(x_0)\cap\partial\Omega_i$$
$$\Gamma_i^*(x_0)=\Gamma^*(x_0)\cap\partial\Omega_i \tag{3.4}$$
$$\Sigma_i(x_0)=\Gamma_i(x_0)\times(0,T), \qquad i=1,2. \tag{3.5}$$

We introduce the function space

$$\begin{aligned}V_{\Gamma^*(x_0)}=\big\{(u_1^0,u_2^0,u_1^1,u_2^1)\in H^1(\Omega_1)\times H^1(\Omega_2)&\\ \times L^2(\Omega_1)\times L^2(\Omega_2)\colon u_1^0=u_2^0 \text{ on } \Gamma_{tr}&\\ \text{and } u_i^0=0 \text{ on } \Gamma_i^*(x_0),\ i=1,2\big\}.&\end{aligned} \tag{3.6}$$

Let us consider the problem of exact controllability

$$y_i''-a_i\Delta y_i=h_i\chi_{\omega_i} \quad \text{in } \Omega_i\times(0,T), \tag{3.7}$$
$$y_i=0 \quad \text{on } \Gamma_i^*(x_0)\times(0,T), \tag{3.8}$$
$$y_i=\phi_i \quad \text{on } \Gamma_i(x_0)\times(0,T), \tag{3.9}$$
$$y_1=y_2,\ \ a_1\frac{\partial y_1}{\partial\nu}=a_2\frac{\partial y_2}{\partial\nu} \quad \text{on } \Gamma_{tr}\times(0,T), \tag{3.10}$$
$$y_i(0)=y_i^0,\ \ y_i'(0)=y_i^1 \quad \text{in } \Omega_i,\ i=1,2, \tag{3.11}$$

where h_i and ϕ_i are controls to be found and χ_{ω_i} denotes the characteristic function of ω_i. We have the following exact controllability theorem.

Theorem 3.1: Let $a=\min\{a_1,a_2\}$ and $T>2R(x_0)/\sqrt{a}$. Then for any

$$(y_1^1,y_2^1,y_1^0,y_2^0)\in\big(V_{\Gamma^*(x_0)}\big)' \quad (\text{the dual space of } V_{\Gamma^*(x_0)})$$

there exist controls $\phi_i\in L^2(\Sigma_i(x_0))$ supported on $\Gamma_i(x_0)$ and $(h_1,h_2)\in C([0,T];(H^1(\Omega_1)\times H^1(\Omega_2))')$ supported on ω such that the solution of (3.7)–(3.11) satisfying

$$y_1(T)=y_2(T)=y_1'(T)=y_2'(T)=0. \tag{3.12}$$

By the Hilbert uniqueness method (HUM) (see, e.g., [7]), it is well known that the above exact controllability is equivalent to the following observability.

Theorem 3.2: Let $a = \min\{a_1, a_2\}$ and $T > 2R(x_0)/\sqrt{a}$. Then there exist a smooth function $\varphi = \varphi(x)$ supported on ω and a positive constant $C > 0$ such that

$$\sum_{i=1}^{2}\left(\int_0^T\int_{\Gamma_i(x_0)}\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma\,dt + \int_0^T\int_{\Omega_i}\varphi(|u_i|^2+|\nabla u_i|^2)\,dx\,dt\right) \ge CE(0) \quad (3.13)$$

for all solutions of (1.1)–(1.4).

Proof: Noting that $u_i = 0$ on $\Gamma\cap\partial\Omega_i$, we deduce from (2.24) and (2.34) that

$$\begin{aligned}\int_\delta^{T-\delta} E(t)dt + \sum_{i=1}^{2}\left(u_i'(t),\, \frac{n-1}{2}\,u_i(t) + m_k\,\frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta} &= \sum_{i=1}^{2}\frac{a_i}{2}\int_\delta^{T-\delta}\int_{\Gamma\cap\partial\Omega_i} m\cdot\nu\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma\,dt\\ &\quad+\int_{\Sigma_{tr}}\left(a_1\frac{\partial u_1}{\partial\nu}m_k\frac{\partial u_1}{\partial x_k} - a_2\frac{\partial u_2}{\partial\nu}m_k\frac{\partial u_2}{\partial x_k}\right.\\ &\qquad\left.+\frac{a_2}{2}m_k\nu_k|\nabla u_2|^2 - \frac{a_1}{2}m_k\nu_k|\nabla u_1|^2\right)d\Sigma.\end{aligned} \quad (3.14)$$

As usual (see, e.g., [7, p. 375, (4.25)]), we can estimate

$$\left|\sum_{i=1}^{2}\left(u_i'(t),\, \frac{n-1}{2}\,u_i(t) + m_k\,\frac{\partial u_i(t)}{\partial x_k}\right)\Bigg|_\delta^{T-\delta}\right| \le \frac{2R(x_0)}{\sqrt{a}}\,E(0). \quad (3.15)$$

Since $E(t)\equiv E(0)$, it therefore follows from (2.29) that:

$$\begin{aligned}(T-2\delta)E(0) - \frac{2R(x_0)}{\sqrt{a}}\,E(0) &\le \sum_{i=1}^{2}\frac{a_i}{2}\int_\delta^{T-\delta}\int_{\Gamma_i(x_0)} m\cdot\nu\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma\,dt\\ &\quad+C\sum_{i=1}^{2}\int_{\omega_i'}\left(|u_i'(\delta)|^2 + |u_i'(T-\delta)|^2 + |\nabla u_i(\delta)|^2\right.\\ &\qquad\left.+|\nabla u_i(T-\delta)|^2\right)dx\\ &\quad+C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega_i'}|u_i'|^2\,dx\,dt\\ &\quad+C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega_i'}|\nabla u_i|^2\,dx\,dt\end{aligned} \quad (3.16)$$

where ω' is the subdomain given in (2.28). We now estimate the second term of the right-hand side. Let ω' and ω'' be given in (2.28) and θ be the smooth function given in (2.30). Multiplying (1.1) by $\theta u_i'$ and integrating over Ω_i, we obtain

$$\sum_{i=1}^{2}\frac{d}{dt}\int_{\Omega_i}\theta\left(|u_i'|^2 + a_i|\nabla u_i|^2\right)dx = -2\sum_{i=1}^{2}\int_{\Omega_i}a_iu_i'\nabla\theta\cdot\nabla u_i\,dx \quad (3.17)$$

and then

$$\begin{aligned}\sum_{i=1}^{2}\int_{\Omega_i}\theta\left(|u_i'(s)|^2 + a_i|\nabla u_i(s)|^2\right)dx &- \sum_{i=1}^{2}\int_{\Omega_i}\theta\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\\ &= -2\sum_{i=1}^{2}\int_t^s\int_{\Omega_i}a_iu_i'(\tau)\nabla\theta\cdot\nabla u_i(\tau)\,dx\,d\tau.\end{aligned} \quad (3.18)$$

Taking $s = \delta$ in (3.18), we obtain

$$\begin{aligned}(T-2\delta)\sum_{i=1}^{2}\int_{\omega'}\left(|u_i'(\delta)|^2 + a_i|\nabla u_i(\delta)|^2\right)dx &= \sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega'}\left(|u_i'(\delta)|^2 + a_i|\nabla u_i(\delta)|^2\right)dx\,dt\\ &\le \sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\Omega_i}\theta\left(|u_i'(\delta)|^2 + a_i|\nabla u_i(\delta)|^2\right)dx\,dt\\ &\le \sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\Omega_i}\theta\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\,dt\\ &\quad+2\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_\delta^t\int_{\Omega_i}a_i|u_i'(s)\nabla\theta\cdot\nabla u_i(s)|\,dx\,ds\,dt\\ &\le \sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega''}\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\,dt\\ &\quad+C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_\delta^{T-\delta}\int_{\omega''}\\ &\quad\times\left(|u_i'(s)|^2 + a_i|\nabla u_i(s)|^2\right)dx\,ds\,dt\\ &\le C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega''}\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\,dt.\end{aligned} \quad (3.19)$$

Similarly, by taking $s = T-\delta$ in (3.18), we deduce

$$\begin{aligned}(T-2\delta)\sum_{i=1}^{2}\int_{\omega'}\left(|u_i'(T-\delta)|^2 + a_i|\nabla u_i(T-\delta)|^2\right)dx &\le C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega''}\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\,dt.\end{aligned} \quad (3.20)$$

It therefore follows from (3.16), (3.19), and (3.20) that:

$$\begin{aligned}E(0) &\le C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\Gamma_i(x_0)}\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma\,dt\\ &\quad+C\sum_{i=1}^{2}\int_\delta^{T-\delta}\int_{\omega''}\left(|u_i'(t)|^2 + a_i|\nabla u_i(t)|^2\right)dx\,dt.\end{aligned} \quad (3.21)$$

On the other hand, let ω''' be a open set such that $\omega''\subset\omega'''\subset\omega$ and take

$$\rho = (t-\delta_1)^2(T-\delta_1-t)^2\psi(x) \quad (3.22)$$

where $0 < \delta_1 < \delta$ and $\psi \in C^\infty(\Omega)$ is a nonnegative function such that $\psi = 1$ on ω'' and $\psi = 0$ on $\Omega - \omega'''$. Set

$$m_1 = \min_{\delta \le t \le T-\delta} \left\{ (t-\delta_1)^2 (T-\delta_1-t)^2 \right\} > 0. \tag{3.23}$$

Multiplying (1.1) by ρu_i and integrating over $(\delta_1, T-\delta_1) \times \Omega_i$, we obtain

$$\sum_{i=1}^{2} \int_{\delta_1}^{T-\delta_1} \int_{\Omega_i} \rho |u_i'|^2 \, dx \, dt = \sum_{i=1}^{2} \int_{\delta_1}^{T-\delta_1} \int_{\Omega_i} \left(a_i |\nabla u_i|^2 \rho + a_i \nabla u_i \cdot \nabla \rho u_i + \tfrac{1}{2} u_i^2 \rho'' \right) dx \, dt. \tag{3.24}$$

It therefore follows that:

$$\begin{aligned} \sum_{i=1}^{2} \int_{\delta}^{T-\delta} \int_{\omega''} |u_i'|^2 \, dx \, dt &\le m_1^{-1} \sum_{i=1}^{2} \int_{\delta}^{T-\delta} \int_{\Omega_i} \rho |u_i'|^2 \, dx \, dt \\ &\le m_1^{-1} \sum_{i=1}^{2} \int_{\delta_1}^{T-\delta_1} \int_{\Omega_i} \rho |u_i'|^2 \, dx \, dt \\ &\le C \sum_{i=1}^{2} \int_{\delta_1}^{T-\delta_1} \int_{\omega'''} \left(|u_i|^2 + |\nabla u_i|^2 \right) dx \, dt. \end{aligned} \tag{3.25}$$

Hence, by (3.21) and (3.25), we deduce that

$$\begin{aligned} E(0) \le C \sum_{i=1}^{2} \int_0^T \int_{\Gamma_i(x_0)} \left| \frac{\partial u_i}{\partial \nu} \right|^2 d\Gamma \, dt \\ + C \sum_{i=1}^{2} \int_0^T \int_{\omega'''} \left(|u_i|^2 + |\nabla u_i|^2 \right) dx \, dt. \end{aligned} \tag{3.26}$$

By taking a nonnegative function $\varphi \in C^\infty(\Omega)$ such that $\varphi = 1$ on ω''' and $\varphi = 0$ on $\Omega - \omega$, we deduce (3.13). $\square$

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Weijiu Liu received the B.S. degree from the Department of Mathematics of the Hunan Normal University, China, and the Ph.D. degree from the Department of Mathematics and Applied Statistics of the University of Wollongong, Australia, in 1992 and 1998, respectively.

He worked as a Postdoctoral Fellow in the Department of Mechanical and Aerospace Engineering of the University of California, San Diego, and the Department of Mathematics and Statistics of the Dalhousie University, Canada, teaching differential equations and doing research in control theory of infinite dimensional dynamic systems.